This month’s column is going to look at a simple model of a vortex called the Rankine vortex. The model is extraordinarily simple and it is somewhat astonishing how well it matches results from real world observations. [Giaiotti and Stel](https://moodle2.units.it/pluginfile.php/21382/mod_resource/content/1/rankine-vortex-notes.pdf) provide vivid comparisons between the Doppler observations of tornado’s radial velocity distribution and the agreement is strong. This, despite the fact, that the Rankine vortex is really an ideal fluid model that doesn’t carry over into the realm of the Navier-Stokes equations where viscosity is important.

The presentation here follows David Acheson’s textbook Elementary Fluid Dynamics with some inspiration and direction from a lecture by Luca Brandt on the Rankine Vortex available [here](https://www.mech.kth.se/~luca/Smak/rec10.pdf).

The Rankine vortex consists of two parts: an inner core that moves as a rigid body and a vortex free exterior. Mathematically, the Rankine vortex is specified as

\[ R = \left\{ \begin{array}{c} \Omega r \;\; r < a \\ \frac{\Omega a^2}{r} \; \; r >a \end{array} \right. \, \]

with $$u\_r = u\_z = 0$$.

By its geometry, the Rankine vortex demands us to work in cylindrical coordinates. There are two steps that one must pay attention to in order to setup the problem. First, by standard vector calculus results, the convective derivative is given by

\[ {\vec u} \cdot \nabla = u\_r \partial\_r + \frac{u\_{\theta}}{r} \partial\_\theta + u\_z \partial\_z \; .\]

Second, the fluid velocity, given in terms of the radial, azimuthal, and vertical unit vectors, is

\[ {\vec u} = u\_r {\hat e}\_r + u\_\theta {\hat e}\_\theta + u\_z {\hat e}\_z \; ,\]

with the realization that the radial and azimuthal unit vectors change as a function of space according to

\[ \partial\_\theta {\hat e\_r} = {\hat e}\_\theta \]

and

\[ \partial\_\theta {\hat e\_\theta} = -{\hat e}\_r \; .\].

Now applying the convective derivative to the velocity,

\[ (\vec u \cdot \nabla) \left[ u\_r {\hat e}\_r + u\_\theta {\hat e}\_\theta + u\_z {\hat e}\_z \right] \; ,\]

formally, gives a lot of terms

\[ (\vec u \cdot \nabla) {\vec u} = \left[ u\_r \partial\_r u\_r + \frac{u\_\theta}{r} \partial\_\theta u\_r + u\_z \partial\_z u\_r - \frac{u\_\theta^2}{r} \right] {\hat e\_r} + \left[ u\_r \partial\_r u\_\theta + \frac{u\_r u\_\theta}{r} + \frac{u\_\theta}{r} \partial\_\theta u\_\theta + u\_z \partial\_z u\_\theta \right] {\hat e}\_\theta + \left[ u\_r \partial\_r u\_z + \frac{u\_\theta}{r} \partial\_\theta u\_z + u\_z \partial\_z u\_z \right] {\hat e}\_z \; ,\]

but, by the assumptions in the Rankine vortex, most are zero. Putting all these pieces together results in the convective derivative taking the form

\[ \left( \vec u \cdot \nabla \right) \vec u = -\frac{u\_\theta^2}{r} {\hat e}\_r \; . \]

Given that the flow is steady, the resulting model is the following partial, coupled, differential equations

\[ \rho –\frac{u\_\theta^2}{r} {\hat e}\_r = -\frac{\partial P}{\partial r} {\hat e}\_r \; , \]

\[ 0 = - \frac{\partial P}{\partial \theta} {\hat e}\_\theta \; , \]

and

\[ 0 = \left( -\frac{\partial P}{\partial z} - \rho g \right) {\hat e}\_z \; . \]

The $$\theta$$ equation merely tells us that $$P = P(r,z)$$. The $$z$$ equation is easily solved for both the inner and outer regions of the vortex because there is no discontinuity. This gives

\[ P(r,z) = - \rho g z + f(r) \; .\]

The radial profile requires a bit more work as the inner and outer regions have to have separate integrations. For the inner

\[ \frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{ \Omega^2 r^2}{r} = \Omega^2 r \; r < a \; . \]

In similar fashion, the outer integration yields

\[ \frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{\frac{\Omega^2 a^r}{r^2}}{r} = \frac{\Omega^2 a^4}{r^3} \; r > a \; . \]

Solving both of these equations gives

\[ P(r,z) = \left\{ \begin{array}{c} \frac{1}{2} \rho \Omega^2 r^2 - \rho g z + C\_1 \; r < a \\ - \frac{1}{2} \rho \frac{\Omega^2 a^4}{r^2} - \rho g z + C\_2 \; r > a \end{array} \right. \]

Continuity at $$r=a$$ gives

\[ -\frac{1}{2} \rho \Omega^2 a^2 - \rho g z + C\_2 = \frac{1}{2} \rho \Omega^2 a^2 - \rho g z + C\_1 \; , \]

which leads to

\[ C\_2 – C\_1 = \rho \Omega^2 a^2 \; .\]

We can determine the pressure difference between the core and far from the core by interrogating the above result. The pressure at the core is

\[ P(0,z) = -\rho g z + C\_1 \; \]

and the pressure at infinity is

\[ P(\infity,z) = - \rho g z + C\_2 \; .\]

Their difference is

\[ P(\infity,z) – P(0,z) = C\_2 – C\_1 = \rho \Omega^2 a^2 \; , \]

which explains why the pressure at the center of a tornado (vortex) is lower than the pressure far from the core.

The free surface (the shape) of the vortex as a function of radius is found by setting $$P(r,z) = P\_0$$ with the height being assumed to be $$z\_0$$ at $$r=0$$.